Title
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Journal
Geometriae Dedicata, 207(1)

ISSN
0046-5755

Authors
Hirsch, MW
Turiel, FJ

Publication Date
2020-08-01

DOI
10.1007/s10711-019-00497-3

Peer reviewed

PRIMARY SINGULARITIES OF VECTOR FIELDS ON SURFACES

M.W. HIRSCH AND F.J. TURIEL

ABSTRACT. Unless another thing is stated one works in the C^∞ category and manifolds have empty boundary. Let X and Y be vector fields on a manifold M. We say that Y tracks X if $[Y, X] = fX$ for some continuous function $f: M \to \mathbb{R}$. A subset K of the zero set $\mathsf{Z}(X)$ is an essential block for X if it is non-empty, compact, open in $\mathsf{Z}(X)$ and its Poincaré-Hopf index does not vanishes. One says that X is non-flat at p if its ∞-jet at p is non-trivial. A point p of $\mathsf{Z}(X)$ is called a primary singularity of X if any vector field defined about p and tracking X vanishes at p. This is our main result: Consider an essential block K of a vector field X defined on a surface M. Assume that X is non-flat at every point of K. Then K contains a primary singularity of X. As a consequence, if M is a compact surface with non-zero characteristic and X is nowhere flat, then there exists a primary singularity of X.

CONTENTS

1. INTRODUCTION

Whether a family of vector fields has a common singularity is a classical issue in dynamical systems. For instance, on a compact surface with non-vanishing Euler characteristic there always exists a common zero provided that the vector fields commute (Lima [9]) or if they span a finite-dimensional nilpotent Lie algebra (Plante [10]). On the existence of a common singularity for a family of commuting vector fields in dimension ≥ 3 several interesting results are due to Bonatti [2] (analitic in dimension 3 and 4) and Bonatti & De Santiago [3] (dimension 3). For a complementary discussion on the existence of a common zero the reader is referred to the introduction of [6].

In this paper one shows that on surfaces every essential block of a nowhere flat vector field X includes a point at which all vector fields tracking X vanish (see Theorem 1.1 below).

Throughout this work manifolds (without boundary) and their associated objects are real C^∞ unless another thing is stated. Consider a tensor $\mathcal{T}$ on a manifold P. Given $p \in P$ the *principal part* of $\mathcal{T}$ at p means $j_p^n\mathcal{T}$ if $j_p^{n-1}\mathcal{T} = 0$ but $j_p^n\mathcal{T} \neq 0$, or zero if $j_p^\infty\mathcal{T} = 0$. The *order* of $\mathcal{T}$ at p is n in the first case and ∞ in the second one. One will say that $\mathcal{T}$ is *flat at p* if its order at this point equals ∞, and *non-flat* otherwise.

In coordinates about p the principal part is identified to the first significant term of the Taylor expansion of $\mathcal{T}$ at p. Given a function f such that $f(p) \neq 0$, the principal part of $f\mathcal{T}$ at p equals that of $\mathcal{T}$ multiplied by $f(p)$.

$\mathsf{Z}(\mathcal{T})$ denotes the set of zeros of $\mathcal{T}$ and $\mathsf{Z}_n(\mathcal{T})$, where $n \in \mathbb{N}'$ and $\mathbb{N}' := \mathbb{N} \cup \{\infty\}$, the set of zeros of order n. (Here $\mathbb{N}$ is the set of positive integers.) Notice that $\mathsf{Z}(\mathcal{T}) = \bigcup_{k\in\mathbb{N}'} \mathsf{Z}_n(\mathcal{T})$ where the union is disjoint.

Consider a vector field Y on P. Y *tracks* $\mathcal{T}$ provided $L_Y\mathcal{T} = f\mathcal{T}$ for some continuous function $f\colon P \to \mathbb{R}$, referred to as the *tracking function*. (When $\mathcal{T}$ is also a vector field this means $[Y, \mathcal{T}] = f\mathcal{T}$.) A set $\mathcal{A}$ of vector fields on P tracks $\mathcal{T}$ provided each element of $\mathcal{A}$ tracks X.

A point $p \in \mathsf{Z}(\mathcal{T})$ is a *primary singularity* of $\mathcal{T}$ if every vector field defined about p that tracks $\mathcal{T}$ vanishes at p. Obviously isolated singularities are primary. The notion of primary singularity is the fundamental new concept of this work.

Let X be a vector field on P. Consider an open set U of P with compact closure $\overline{U}$ such that $\mathsf{Z}(X) \cap (\overline{U} \setminus U) = \emptyset$. The *index* of X on U, denoted by $\mathsf{i}(X, U) \in \mathbb{Z}$, is defined as the Poincaré-Hopf index of any sufficiently close approximation X' to $X|\overline{U}$ (in the compact open topology) such that $\mathsf{Z}(X')$ is finite. Equivalently: $\mathsf{i}(X, U)$ is the intersection number of $X|U$ with the zero section of the tangent bundle (Bonatti [2]). This number is independent of the approximation, and is stable under perturbation of X and replacement of U by smaller open sets containing $\mathsf{Z}(X) \cap U$.

A compact set $K \subset \mathsf{Z}(X)$ is a *block* of zeros for X (or an *X-block*) provided K is non-empty and relatively open in $\mathsf{Z}(X)$, that is to say provided K is non-empty and $\mathsf{Z}(X) \setminus K$ is closed in P. Observe that a non-empty compact $K \subset \mathsf{Z}(X)$ is a X-block if and only if it has a precompact open neighborhood $U \subset P$, called *isolating* for (X, K), such that $\mathsf{Z}(X) \cap \overline{U} = K$ (manifolds are normal spaces). This implies $\mathsf{i}(X, U)$ is determined by X and K, and does not depend on the choice of U. The *index of X at K* is $\mathsf{i}_K(X) := \mathsf{i}(X, U)$. The X-block K is *essential* provided $\mathsf{i}_K(X) \neq 0$, which implies $K \neq \varnothing$, and *inessential* otherwise.

If P is compact, it is isolating for every vector field on P and its set of zeros. Therefore, in this case, $\mathsf{i}_{\mathsf{Z}(X)}(X) = \mathsf{i}(X, P) = \chi(P)$.

This is our main result, which will be proved in the Section 2.1.

Theorem 1.1. *Consider an essential block K of a vector field X defined on a surface M. Assume that X is non-flat at every point of K. Then K contains a primary singularity of X.*

As a straightforward consequence:

Corollary 1.2. *On a compact connected surface M with $\chi(M) \neq 0$ consider a vector field X. Assume that X is nowhere flat. Then there exists a primary singularity of X.*

Moreover, four examples illustrating these results are given in Section 3.

Remark 1.3.

(a) The hypothesis on the non-flatness of Theorem 1.1 and Corollary 1.2 cannot be omitted as the following example shows. On $S^2 \subset \mathbb{R}^3$ consider the vector field $X = \varphi(x_3)(-x_2\partial/\partial x_1 + x_1\partial/\partial x_2)$ where $\varphi(0) = 1$ and $\varphi(\mathbb{R} \setminus (-1/2, 1/2)) = 0$. Then the vector fields $Y = -x_2\partial/\partial x_1 + x_1\partial/\partial x_2$ and $V = \psi(x_3)(-x_3\partial/\partial x_1 + x_1\partial/\partial x_3)$ where $\psi(1) = \psi(-1) = 1$ and $\psi([-3/4, 3/4]) = 0$ track X and $\mathsf{Z}(Y) \cap \mathsf{Z}(V) = \emptyset$. Therefore X has no primary singularity.

(b) Two particular cases of Theorem 1.1 were already known, namely: if X and K are as in the foregoing theorem and $\mathcal{G}$ is a finite-dimensional Lie algebra of vector fields on M that tracks X, then the the elements of $\mathcal{G}$ have a common singularity in K provided that $\mathcal{G}$ is supersolvable (Theorem 1.4 of [5]) or $\mathcal{G}$ and X are analytic (real case of Theorem 1.1 of [6]). Thus these two results are generalized here.

For general questions on Differential Geometry readers are referred to [8], and for those on Differential Topology to [4].

2. Other results

One will need:

Lemma 2.1. *On a manifold P of dimension $m \geq 1$ consider a vector field X of finite order $n \geq 1$ at a point p. Then for almost every $v \in T_pP$ there exists a vector field U defined around p such that $U(p) = v$ and the n-times iterated bracket $[U, [U, \dots [U, X] \dots]]$ does not vanish at p.*

Proof. It suffices to prove the result for $0 \in \mathbb{R}^m$ and a non-vanishing n-homogeneous polynomial vector field $X = \sum_{\ell=1}^m Q_\ell\partial/\partial x_\ell$. Up to a change of the order of the coordinates, we may suppose $Q_1 \neq 0$.

Given $a = (a_1, \dots, a_m) \in \mathbb{R}^m$ set $U_a\colon = \sum_{\ell=1}^m a_\ell \partial/\partial x_\ell$. It suffices to show that for almost any $a \in \mathbb{R}^m - \{0\}$ one has $(U_a \cdots U_a \cdot Q_1)(0) \neq 0$, which is equivalent to show that the restriction of Q_1 to the vector line spanned by a does not vanish identically. But this last assertion is obvious. □

Given a vector field V on a manifold P, a set $S \subset P$ is *V-invariant* if it contains the orbits under V of its points.

Proposition 2.2. *Consider two vector fields X, Y on a surface M. Assume that Y tracks X with tracking function f. Then each set $\mathsf{Z}_n(X)$, $n \in \mathbb{N}'$, is Y-invariant.*

Moreover f is differentiable on the open set

$$[M \setminus \mathsf{Z}(X)] \cup [(\mathsf{Z}(X) \setminus \mathsf{Z}_\infty(X)) \cap (M \setminus \mathsf{Z}(Y))].$$

This result is a consequence of the following two lemmas.

Lemma 2.3. *Under the hypotheses of Proposition 2.2 consider $p \in \mathsf{Z}_n(X)$, $n < \infty$, such that $Y(p) \neq 0$. One has:*

(a) *f is differentiable around p.*
(b) *Let $\gamma\colon (a, b) \to M$ be an integral curve of Y with $\gamma(t_0) = p$ for some $t_0 \in (a, b)$. Then there exists $\varepsilon > 0$ such that $\gamma(t_0 - \varepsilon, t_0 + \varepsilon) \subset \mathsf{Z}_n(X)$.*

Proof. Around p consider a vector field U as in Lemma 2.1 such that $U(p), Y(p)$ are linearly independent. Then there are coordinates (x_1, x_2) about $p \equiv 0$, whose domain D can be identified to a product of two open intervals $J_1 \times J_2$, such that $Y = \partial/\partial x_1$ and $U = \partial/\partial x_2 + x_1 V$.

Let $X = g_1 \partial/\partial x_1 + g_2 \partial/\partial x_2$. Then

$$\frac{\partial g_k}{\partial x_1} = f g_k,\ k = 1, 2.$$

Since f is continuous the general solution to the equation above is:

$$g_k(x) = h_k(x_2) e^{\varphi},\ k = 1, 2,$$

where $\partial\varphi/\partial x_1 = f$ and $\varphi(\{0\} \times J_2) = 0$.

From the Taylor expansion at p of X and U it follows that

$$[U,[U,\dots[U,X]\dots]](0)=\left[\frac{\partial}{\partial x_2},\left[\frac{\partial}{\partial x_2},\dots\left[\frac{\partial}{\partial x_2},X\right]\dots\right]\right](0)$$

for the n-times iterated bracket.

Note that

$$\left[\frac{\partial}{\partial x_2},\left[\frac{\partial}{\partial x_2},\dots\left[\frac{\partial}{\partial x_2},X\right]\dots\right]\right](0)=\frac{\partial^n g_1}{\partial x_2^n}(0)\frac{\partial}{\partial x_1}+\frac{\partial^n g_2}{\partial x_2^n}(0)\frac{\partial}{\partial x_2}.$$

Since on $\{0\}\times J_2$ each $g_k=h_k$ finally one has

$$\frac{\partial^n h_1}{\partial x_2^n}(0)\frac{\partial}{\partial x_1}+\frac{\partial^n h_2}{\partial x_2^n}(0)\frac{\partial}{\partial x_2}=[U,[U,\dots[U,X]\dots]](0)\neq 0,$$

which implies the existence of two diferentiable functions $\tilde{h}_1(x_2)$ and $\tilde{h}_2(x_2)$ such that $h_k=x_2^n\tilde{h}_k(x_2)$, $k=1,2$, and $\tilde{h}_1^2(0)+\tilde{h}_2^2(0)>0$.

Therefore by shrinking D if necessary, we may suppose that at least one of these function, say $\tilde{h}_\ell$, does not have any zero. Observe that f will be differentiable if $\tilde{h}_\ell e^\varphi$ is differentiable because $\tilde{h}_\ell$ is differentiable without zeros and $\partial\varphi/\partial x_1=f$.

As $g_\ell=x_2^n\cdot(\tilde{h}_\ell e^\varphi)$, it follows that g_ℓ is divisible by $1,x_2,\dots,x_2^n$ and the respective quotient functions are at least continuous. Moreover g_ℓ/x^r, $r=1,\dots,n-1$, vanish if $x_2=0$, that is to say on $J_1\times\{0\}$.

The Taylor expansion of g_ℓ transversely to $J_1\times\{0\}$ leads

$$g_\ell=\sum_{r=0}^{n-1}x_2^r\mu_r(x_1)+x_2^n\mu_n(x_1,x_2)$$

where each μ_k, $k=1,\dots,n$ is differentiable.

Now since $g_\ell(J_1\times\{0\})=0$ one has $\mu_0=0$.

In turn as g_ℓ/x_2 equals zero on $J_1\times\{0\}$ it follows $\mu_1=0$, and so one. Hence $\mu_0=\cdots=\mu_{n-1}=0$, which implies $g_\ell=x_2^n\mu_n(x_1,x_2)$. Therefore $\tilde{h}_\ell e^\varphi=\mu_n$ is differentiable, which proves (a).

On the other hand, as e^φ is differentiable and positive, X and

$$X':=e^{-\varphi}X=x_2^n\left(\tilde{h}_1\frac{\partial}{\partial x_1}+\tilde{h}_2\frac{\partial}{\partial x_2}\right)$$

have the same order everywhere. Thus X has order n at every point of $J_1\times\{0\}$ and (b) becomes obvious. $\square$

Lemma 2.4. *Under the hypotheses of Proposition 2.2 consider $p \in \mathsf{Z}_\infty(X)$ with $Y(p) \neq 0$. Let $\gamma\colon (a,b) \to M$ be an integral curve of Y passing through p for some $t_0 \in (a,b)$. Then there exists $\varepsilon > 0$ such that $\gamma(t_0-\varepsilon, t_0+\varepsilon) \subset \mathsf{Z}_\infty(X)$.*

Proof. Around $p \equiv 0$ consider coordinates (y_1, y_2), whose domain E can be identified to a product of two open intervals $K_1\times K_2$, such that $Y = \partial/\partial y_1$ and $X = a_1(y_2)e^\rho\partial/\partial y_1 + a_2(y_2)e^\rho\partial/\partial y_2$ where $\partial\rho/\partial y_1 = f$ and $\rho(\{0\}\times K_2) = 0$. These coordinates exist by the same reason as in the proof of Lemma 2.3.

Assume the existence of a $q \in K_1 \times \{0\}$ of finite order n.

Since $p \in \mathsf{Z}_\infty$ and e^ρ equals 1 on $\{0\}\times K_2$, it follows that $j_0^\infty a_1 = j_0^\infty a_2 = 0$. Therefore $a_k(y_2) = y_2^{n+1}b_k(y_2)$, $k = 1,2$, where each b_k is differentiable. Hence there exists a continuous vector field X_n such that $X = y_2^{n+1}X_n$; that is to say X is continuously divisible by y_2^{n+1}.

In turn one can find coordinates (x_1, x_2) around $q \equiv 0$ whose domain D can be identify to $J_1 \times J_2$ as in the proof of Lemma 2.3, which implies that

$$X = x_2^n e^\varphi \left(\tilde{h}_1(x_2)\frac{\partial}{\partial x_1} + \tilde{h}_2(x_2)\frac{\partial}{\partial x_2}\right)$$

where $\tilde{h}_1\partial/\partial x_1 + \tilde{h}_2\partial/\partial x_2$ has no zero on D.

By shrinking D if necessary, we may suppose $D \subset E$. Then, regarded both sets in M, $J_1 \times \{0\}$ is a subset of $K_1 \times \{0\}$ since they are traces of integral curves of Y with q as common point.

On the other hand as y_2 vanishes on $K_1\times\{0\}$ but its derivative never does, on D one has $y_2 = x_2 c(x_1, x_2)$ where c has no zero. This fact implies that X on D is continuously divisible by x_2^{n+1} because it was continuously divisible by y_2^{n+1}.

But clearly from the expression of X in coordinates (x_1, x_2) it follows the non-divisibility by x_2^{n+1}, contradiction. In short the order of X at each point of $K_1 \times \{0\}$ is infinite. $\square$

Remark 2.5. Under the hypotheses of Proposition 2.2 the tracking function f can be not differentiable around a flat point. For instance, on $\mathbb{R}^2$ set $Y = x_1^4\partial/x_1 + \partial/\partial x_2$ and $X = g(x_1)\partial/x_1$, where $g(x_1) = e^{-1/x_1}$ if $x_1 > 0$, $g(x_1) = e^{-1/x_1^2}$ if $x_1 < 0$ and $g(0) = 0$. Then $f(x) = x_1^2 - 4x_1^3$ if $x_1 > 0$, $f(x) = 2x_1 - 4x_1^3$ if $x_1 < 0$ and $f(\{0\}\times\mathbb{R}) = 0$, which is not differentiable on $\{0\}\times\mathbb{R}$.

Proof of Proposition 2.2. Let us proves the first assertion. Consider a non-constant integral curve of Y (the constant case is clear) $\gamma\colon (a,b) \to M$. By Lemmas 2.3 and 2.4, $\gamma^{-1}(\mathsf{Z}(X))$ is open in (a,b). As this set is closed too one has $\gamma^{-1}(\mathsf{Z}(X)) = \emptyset$ or $\gamma^{-1}(\mathsf{Z}(X)) = (a,b)$. The first case is obvious; in the second one $(a,b) = \bigcup_{n\in\mathbb{N}'}\gamma^{-1}(\mathsf{Z}_n(X))$ where each term of this union is open. Therefore a single term of this disjoint union is non-empty since (a,b) is connected.

For the second assertion apply (a) of Lemma 2.3 taking into account that f is always differentiable on $M \setminus \mathsf{Z}(X)$ because, on this set, the quotient $[Y,X]/X$ has a meaning. □

Proposition 2.6. *On a surface M consider a vector field X such that $\mathsf{Z}(X) \neq \emptyset$ but $\mathsf{Z}_\infty(X) = \emptyset$. Then at least one of the following assertions holds:*

(1) $\mathsf{Z}(X)$ *is a regular (embedded)* 1*-submanifold.*
(2) *There exists a primary singularity of X.*

Proof. Assume the non-existence of primary singularities.

Consider any $p \in \mathsf{Z}(X)$ and a vector field Y defined around p with $Y(p) \neq 0$ that tracks X. Let U be a second vector field about p as in Lemma 2.1 such that $U(p), Y(p)$ are linearly independent. Then there exist coordinates (x_1, x_2), about $p \equiv 0$, whose domain D can be identified to a product of two open intervals $J_1 \times J_2$ such that $Y = \partial/\partial x_1$ and $U = \partial/\partial x_2 + x_1 V$.

The same reasoning as in the proof of Lemma 2.3 allows to suppose that

$$X = x_2^n e^{\varphi}\left(\tilde{h}_1\frac{\partial}{\partial x_1} + \tilde{h}_2\frac{\partial}{\partial x_2}\right)$$

with $\tilde{h}_1^2 + \tilde{h}_2^2 > 0$ everywhere.

Therefore $\mathsf{Z}(X) \cap D$ is given by the equation $x_2 = 0$, which implies that $\mathsf{Z}(X)$ is a regular 1-submanifold. □

Theorem 2.7. *Consider a vector field X on a surface M. Assume that:*

(a) $\mathsf{Z}_\infty(X) = \emptyset$.

(b) *There is a connected component of $\mathsf{Z}(X)$ that is not included in a single $\mathsf{Z}_n(X)$.*

Then there exists a primary singularity of X.

Proof. Assume there is no primary singularity. By Proposition 2.6, $\mathsf{Z}(X)$ is a regular 1-submanifold of M. By hypothesis there are a connected component C of $\mathsf{Z}(X)$ and two different natural numbers m and n such that C meets $\mathsf{Z}_m(X)$ and $\mathsf{Z}_n(X)$.

As C is a regular 1-submanifold, Proposition 2.2 and Lemma 2.3 imply that each $C \cap \mathsf{Z}_r(X)$, $r \in \mathbb{N}$, is open in C. Therefore C is a disjoint union of a family of non-empty open sets with two or more elements hence not connected, contradiction. □

2.1. **Proof of of Theorem 1.1.** It consists of three steps.

1. Assume that there is no primary singularity in K. From Proposition 2.6 applied to an isolating open set it follows that K is a compact 1-submanifold. Notice that at least one of its connected component is an essential block. Therefore one may suppose that K is diffeomorphic to S^1 and, by shrinking M, that $\mathsf{Z}(X) = K$.

Consider a Riemannian metric g on M. Given $p \in K$ by reasoning as before one can find coordinates (x_1, x_2) such that $p \equiv 0$ and

$$X = x_2^n e^\varphi \left(\tilde{h}_1 \frac{\partial}{\partial x_1} + \tilde{h}_2 \frac{\partial}{\partial x_2} \right)$$

where $\tilde{h}_1 \partial/\partial x_1 + \tilde{h}_2 \partial/\partial x_2$ has no zero. Therefore around p there exists an 1-dimensional vector subbundle $\mathcal{E}$ of the tangent bundle that is orthogonal to X. Such a vector subbundle is unique because clearly it exists and is unique outside K. Thus, gluing together the local constructions gives rise to an 1-dimensional vector subbundle $\mathcal{E}$ of TM that is orthogonal to X.

2. If $\mathcal{E}$ is trivial there exists a nowhere singular vector field V such that $g(V, X) = 0$. Let $\varphi\colon M \to \mathbb{R}$ be a function with a sufficiently narrow compact support such that $\varphi(K) = 1$. Set $X_\delta := X + \delta\varphi V$, $\delta > 0$. Then X_δ approaches X as much as desired and $\mathsf{Z}(X_\delta) = \emptyset$, so K is an inessential block.

3. Now assume that $\mathcal{E}$ is not trivial. There always exists a 2-folding covering space $\pi\colon M' \to M$ such that the pull-back $\mathcal{E}' \subset TM'$ of the vector subbundle $\mathcal{E}$ is trivial.

Consider the vector field X' on M' defined by $\pi_*(X') = X$. Then $\mathsf{Z}(X') = \pi^{-1}(K)$ and X' is nowhere flat. Moreover $\mathcal{E}'$ is orthogonal to X' with respect to the pull-back of g. Now the same reasoning as in the foregoing step shows that $\mathsf{i}_{\mathsf{Z}(X')}(X') = 0$. But clearly $\mathsf{i}_{\mathsf{Z}(X')}(X') = 2\mathsf{i}_K(X)$ and hence K is inessential.

3. Examples

Example 3.1. In this example one shows two facts. First, primary singularities can exist even if the index of X is not definable. Second, being nowhere flat is a weaker hypothesis than being analytic.

Consider a proper closed subset C of $\mathbb{R}$ and a function $\varphi\colon \mathbb{R} \to \mathbb{R}$ such that $\varphi^{-1}(0) = C$. Set $X := x_1^2\partial/\partial x_1 + x_1\varphi(x_2)\partial/\partial x_2$. Then $\mathsf{Z}(X) = \{0\} \times \mathbb{R}$, $\mathsf{Z}_1(X) = \{0\} \times (\mathbb{R} \setminus C)$, $\mathsf{Z}_2(X) = \{0\} \times C$ and $\mathsf{Z}_n(X) = \emptyset$ for $n \neq 1, 2$, so X is nowhere flat. By Theorem 2.7 the vector field X has primary singularities.

More exactly the set S_a of primary singularities of X equals $\{0\} \times (C \setminus \overset{\circ}{C})$. Indeed:

(1) $\varphi(x_2)\partial/\partial x_2$ tracks X and does not vanish on $\{0\} \times (\mathbb{R} \setminus C)$.
(2) $\partial/\partial x_2$ tracks X on $\mathbb{R} \times \overset{\circ}{C}$.

Therefore $S_a \subset \{0\} \times (C \setminus \overset{\circ}{C})$.

Take $p = (0, c) \in \{0\} \times (C \setminus \overset{\circ}{C})$. Assume the existence around this point of a vector field Y with $Y(p) \neq 0$ that tracks X. Them from Proposition 2.2 and Lemma 2.3 it follows the existence of $\varepsilon > 0$ such that the order of X at every point of $\{0\} \times (c - \varepsilon, c + \varepsilon)$ is constant and hence c belongs to the interior of $\mathbb{R} \setminus C$ or to that of C. Therefore $c \notin C \setminus \overset{\circ}{C}$ contradiction.

In short, each element of $\{0\} \times (C \setminus \overset{\circ}{C})$ is a primary singularity and $S_a = \{0\} \times (C \setminus \overset{\circ}{C})$.

Finally observe that if C is a Cantor set, then X is not analytic for any analytic structure on $\mathbb{R}^2$ since $\mathsf{Z}_2(X) = \{0\} \times C$ is never an analytic set.

Example 3.2. In this example one gives a vector field on S^2, which is analytic so with no flat points, whose zero set is a circle just with two primary singularities.

The sphere S^2 can be regarded as the leaves space of the 1-dimensional foliation on $\mathbb{R}^3 \setminus \{0\}$ associated to the vector field $V = \sum_{k=1}^{3} x_k \partial/\partial x_k$, while the canonical projection $\pi\colon \mathbb{R}^3 \setminus \{0\} \to S^2$ is given by $\pi(x) = x / \parallel x \parallel$.

Every linear vector field U' commutes with V and can be projected by π on a vector field U on S^2. Moreover $U(a) = 0$, where $a = (a_1, a_2, a_3) \in S^2$, if and only if a is an eigenvector of U' regarded as an endomorphism of $\mathbb{R}^3$, that is to say if and only if

$$\left[\sum_{k=1}^{3} a_k \frac{\partial}{\partial x_k}, U' \right] = \lambda \sum_{k=1}^{3} a_k \frac{\partial}{\partial x_k}$$

for some scalar λ.

Set $X\colon = \pi_*(x_1 \partial/\partial x_2)$. Then $\mathsf{Z}(X) = \{x \in S^2\colon x_1 = 0\}$ is an essential block of index two since $\chi(S^2) = 2$. By Corollary 1.2 the set S_a of primary singularities of X is not empty.

For determining it consider the vector field $Y\colon = \pi_*(x_3 \partial/\partial x_2)$. Then $[X, Y] = 0$ because $[x_1 \partial/\partial x_2, x_3 \partial/\partial x_2] = 0$. Moreover $\mathsf{Z}(Y) = \{x \in S^2\colon x_3 = 0\}$.

As Y tracks X, the vector field Y is tangent to $\mathsf{Z}(X)$. On the other hand $\mathsf{Z}(X) \cap \mathsf{Z}(Y) = \{(0, 1, 0), (0, -1, 0)\}$, so $S_a \subset \{(0, 1, 0), (0, -1, 0)\}$. Since

$F_*X = X$, where F is the antipodal map, one has $F(S_a) = S_a$ and hence $S_a = \{(0,1,0),(0,-1,0)\}$.

Example 3.3. Let M be a connected compact surface of non-vanishing Euler characteristic. As it is well known, on M there always exist two vector fields X, Y with no common zero such that $[Y,X] = X$ (Lima [9], Plante [10]; see [1, 12] as well). Therefore there is no primary singularity of X, *but there always exists a periodic regular trajectory of Y included in* $\mathsf{Z}_\infty(X)$.

Indeed, by Corollary 1.2 and Proposition 2.2 the set $\mathsf{Z}_\infty(X)$ is non-empty and Y-invariant. Since $\mathsf{Z}_\infty(X)$ is compact, there always exists a minimal set $S \subset \mathsf{Z}_\infty(X)$ of (the action of) Y.

As $\mathsf{Z}(X) \cap \mathsf{Z}(Y) = \emptyset$, a generalization of the Poincaré-Bendixson theorem [11] implies that S is homeomorphic to a circle. In other words, there exists a non-trivial periodic trajectory of Y consisting of flat points of X.

More generally, given a vector field $\widehat{X}$ on M let $\mathcal{A}$ be the real vector space of those vector fields on M that track $\widehat{X}$. Assume that $\mathsf{Z}(\widehat{X}) \neq M$ and $\mathsf{Z}(\widehat{X}) \cap (\bigcap_{V\in\mathcal{A}} \mathsf{Z}(V)) = \emptyset$. Then by Corollary 1.2 the compact set $\mathsf{Z}_\infty(\widehat{X})$ is not empty and contains a minimal set $\widehat{S}$ of $\mathcal{A}$ (more exactly of the group of diffeomorphisms of M spanned by the flows of the elements of $\mathcal{A}$).

Clearly $\widehat{S}$ is not a point. A second generalization of the Poincaré-Bendixson theorem [7] shows that $\widehat{S}$ is homeomorphic to a circle.

Even more, in our case $\widehat{S}$ is a regular 1-submanifold and hence diffeomorphic to a circle. Let us see it. Take $p \in \widehat{S}$; then there is $V \in \mathcal{A}$ with $V(p) \neq 0$. Consider coordinates (x_1, x_2) around $p \equiv 0$ whose domain D is identified in the natural way to a product $(-\varepsilon,\varepsilon)\times(-\varepsilon,\varepsilon)$ such that $V = \partial/\partial x_1$.

Let $\gamma\colon (-\delta,\delta) \to M$ be an integral curve of V with initial condition $\gamma(0) = p$. Then $\gamma(-\delta,\delta) \subset \widehat{S}$. Moreover, if δ is sufficiently small $\gamma(-\delta,\delta)$ is a relatively open subset of $\widehat{S}$. Indeed, $\gamma\colon(-\delta,\delta) \to \widehat{S}$ will be injective so open because $\widehat{S}$ is a 1-dimensional topological manifold (actually S^1). Now by shrinking D and $(-\delta,\delta)$ if necessary, we may suppose that $\gamma(-\delta,\delta) \subset D$, $\delta = \varepsilon$ and $\gamma(t) = (t,0)$. Thus $(-\varepsilon,\varepsilon)\times\{0\} = \gamma(-\delta,\delta)$ is relatively open in $\widehat{S}$

and there exists an open set E of M such that $E \cap \widehat{S} = (-\varepsilon, \varepsilon) \times \{0\}$. Hence $\widehat{S} \cap (D \cap E)$ is defined by the equation $x_2 = 0$ in the system of coordinates $(D \cap E, (x_1, x_2))$.

3.1. **An example from the blowup process.** In this subsection one constructs a homogeneous polynomial vector field on $\mathbb{R}^2$ whose trajectories but a finite number, let us call them *exceptional*, have the origin both as α and β-limit. Then by blowing up the origin one obtains a new vector field on a Moebius band whose number of primary singularities equals half that of exceptional trajectories of the first vector field.

Thus a global property on the trajectories of a vector field becomes a semi-local property on the primary singularities of another vector field.

First some technical facts. Denote by $\widetilde{\mathbb{R}}^2$ the surface obtained by blowing up the origin of $\mathbb{R}^2$ and by $\tilde{p} \colon \widetilde{\mathbb{R}}^2 \to \mathbb{R}^2$ the canonical projection. Recall that $\widetilde{\mathbb{R}}^2$ is a Moebius band. If X is a vector field on $\mathbb{R}^2$ that vanishes at the origin, the blowup process gives rise to a vector field $\widetilde{X}$ on $\widetilde{\mathbb{R}}^2$ such that $\tilde{p}_* \widetilde{X} = X$. When the origin is an isolated singularity of index k and the order of X at this point is ≥ 2, then $\tilde{p}^{-1}(0)$ is a $\widetilde{X}$-block of index $k - 1$.

Now identify $\mathbb{C}$ to $\mathbb{R}^2$ by setting $z = x_1 + ix_2$. Then each complex vector field $z^n \partial/\partial z$, $n \geq 2$, can be considered as a vector field $X_n = P_n \partial/\partial x_1 + Q_n \partial/\partial x_2$ on $\mathbb{R}^2$ where $z^n = (x_1 + ix_2)^n = P_n(x_1, x_2) + iQ_n(x_1, x_2)$. Our purpose will be to show that $\mathsf{Z}(\widetilde{X}_n) = \tilde{p}^{-1}(0)$ contains $n - 1$ primary singularities of $\widetilde{X}_n$. (Recall that the origin is a singularity of X_n of index n and hence $\tilde{p}^{-1}(0)$ is a $\widetilde{X}_n$-block of index $n - 1$.)

3.1.1. $\widetilde{\mathbb{R}}^2$ *from another point of view.* Consider the map $\varphi \colon \mathbb{R} \times S^1 \to \mathbb{R}^2$ given by $\varphi(r, \theta) = (r cos\theta, r sin\theta)$. Then $\varphi \colon \mathbb{R}_+ \times S^1 \to \mathbb{R}^2 \setminus \{0\}$ and $\varphi \colon \mathbb{R}_- \times S^1 \to \mathbb{R}^2 \setminus \{0\}$ are diffeomorphisms, and $\varphi(r, \theta) = \varphi(r', \theta')$ with $(r, \theta), (r', \theta') \in (\mathbb{R} \setminus \{0\}) \times S^1$ if and only if $(r, \theta) = (r', \theta')$ or $(r', \theta') = (-r, \theta + \pi)$.

Let $\sim$ be the equivalence relation on $\mathbb{R} \times S^1$ defined by $(r, \theta) \sim (r', \theta')$ if and only if $(r, \theta) = (r', \theta')$ or $(r', \theta') = (-r, \theta + \pi)$. Then the quotient space $M_s \colon = (\mathbb{R} \times S^1)/\sim$ is a Moebius strip and the canonical projection $p \colon \mathbb{R} \times S^1 \to M_s$ is a (differentiable) covering space with two folds. Moreover the map $\bar{\varphi} \colon M_s \to \mathbb{R}^2$ given by $\bar{\varphi}(p(r, \theta)) = \varphi(r, \theta)$ is well defined and differentiable.

Recall that $\tilde{p}^{-1}(0) = \mathbb{R}P^1$ is the space of vector lines in $\mathbb{R}^2$ and $\tilde{p} \colon \widetilde{\mathbb{R}}^2 \setminus \tilde{p}^{-1}(0) \to \mathbb{R}^2 \setminus \{0\}$ a diffeomorphism. Now one defines $\Psi \colon M_s \to \widetilde{\mathbb{R}}^2$ as follows:

(a) $\Psi(p(r, \theta)) = \tilde{p}^{-1}(\varphi(r, \theta))$ if $r \neq 0$,

(b) $\Psi(p(r, \theta))$ equals the vector line of $\mathbb{R}^2$ spanned by $(cos\theta, sin\theta)$ if $r = 0$.

It is easily checked that $\Psi \colon M_s \to \widetilde{\mathbb{R}}^2$ is a diffeomorphism and $\tilde{p} \circ \Psi = \bar{\varphi}$. Therefore $\tilde{p} \colon \widetilde{\mathbb{R}}^2 \to \mathbb{R}^2$ and $\bar{\varphi} \colon M_s \to \mathbb{R}^2$ can be identified in this way. For sake of simplicity in what follows $\tilde{p} \colon \widetilde{\mathbb{R}}^2 \to \mathbb{R}^2$ will replaced by $\bar{\varphi} \colon M_s \to \mathbb{R}^2$ in our computations. Thus if X is a vector field on $\mathbb{R}^2$ that vanishes at the origin, then $\widetilde{X}$ will be the single vector field on M_s such that $\bar{\varphi}_* \widetilde{X} = X$.

On the other hand X' will denote the pull-back by p of $\widetilde{X}$. Clearly $\varphi_* X' = X$. Moreover with respect to X' the index of $\{0\} \times S^1$ and the number of primary singularities included in it are twice those of $\bar{\varphi}^{-1}(0)$ relative to $\widetilde{X}$.

As a consequence, in the case of X_n it will suffice to show that $\mathsf{Z}(X'_n) = \{0\} \times S^1$ contains $2n - 2$ singularities of X'_n.

3.1.2. *Computation of the primary singularities of X'_n.* As $\varphi \colon (\mathbb{R} \setminus \{0\}) \times S^1 \to \mathbb{R}^2 \setminus \{0\}$ is a covering space any vector field on $\mathbb{R}^2 \setminus \{0\}$ can be lifted up. Denote by $\partial'/\partial x_k$, $k = 1, 2$, the lifted vector field of $\partial/\partial x_k$. Then

$$\frac{\partial'}{\partial x_1} = cos\theta \frac{\partial}{\partial r} - r^{-1} sin\theta \frac{\partial}{\partial \theta} \quad \text{and} \quad \frac{\partial'}{\partial x_2} = sin\theta \frac{\partial}{\partial r} + r^{-1} cos\theta \frac{\partial}{\partial \theta}.$$

Since $(rcos\theta + irsin\theta)^n = r^n cos(n\theta) + ir^n sin(n\theta)$ one has $P_n \circ \varphi = r^n cos(n\theta)$ and $Q_n \circ \varphi = r^n sin(n\theta)$. Observe that on $(\mathbb{R} \setminus \{0\}) \times S^1$ the vector field X'_n is the lifted one of X_n, so $X'_n = r^n cos(n\theta)\partial'/\partial x_1 + r^n sin(n\theta)\partial'/\partial x_2$. Finally, developing the foregoing expression of X'_n and extending it by continuity to $\mathbb{R} \times S^1$ yields:

$$X'_n = r^{n-1}\left(rcos((n-1)\theta)\frac{\partial}{\partial r} + sin((n-1)\theta)\frac{\partial}{\partial \theta}\right)$$

The vector field $Y = rcos((n-1)\theta)\partial/\partial r + sin((n-1)\theta)\partial/\partial\theta$ tracks X'_n with tracking function $(n-1)cos((n-1)\theta)$. Therefore the set S_a of primary singularities of X'_n is included in $\{0\} \times T_n$ where $T_n :\ = \{\theta \in S^1 : sin((n-1)\theta) = 0\}$.

On the other hand, the order of X'_n at the points of $\{0\} \times (S^1 \setminus T_n)$ is $n-1$ and strictly greater than $n-1$ at the points of $\{0\} \times T_n$. As T_n is finite, more exactly it has $2n-2$ elements, Proposition 2.2 and Lemma 2.3 imply that all the points of $\{0\} \times T_n$ are primary singularities. In short $S_a = \{0\} \times T_n$ and hence $\mathsf{Z}(\widetilde{X}_n) = \tilde{p}^{-1}(0)$ contains $n-1$ primary singularities.

3.1.3. *The geometric meaning of the primary singularities of* $\widetilde{X}_n$. When $n \geq 2$ the complex flow of $z^n\partial/\partial z$ is

$$\Phi(z,t) = z\left[(1-n)tz^{n-1} + 1\right]^{\frac{1}{1-n}}$$

with initial condition $\Phi(z, 0) = z$.

(Fixed $z \neq 0$ consider as domain of the variable t the open set $D_z :\ = \mathbb{C} \setminus R_z$ where $R_z :\ = \{s(n-1)^{-1}z^{1-n} : s \in [1, \infty)\}$. Note that D_z is star shaped with respect to the origin. Since D_z is simply connected, the initial condition $\Phi(z, 0) = z$ defines a single continuous and hence holomorphic map $\Phi(z, \quad) : D_z \to \mathbb{C}$. Thus the apparent ambiguity introduced by the root of order $n-1$ is eliminated.)

On the other hand considering, in the foregoing expression of Φ, real values of t only and identifying z with (x_1, x_2) yield the real flow of X_n. Therefore given $(x_1, x_2) \in \mathbb{R}^2 \setminus \{0\}$ if $z^{n-1} = (x_1 + ix_2)^{n-1}$ is not a real number, its X_n-trajectory is defined for any $t \in \mathbb{R}$ and has the origin both as α and ω-limit.

On the contrary when $z^{n-1} = (x_1 + ix_2)^{n-1}$ is a real number, the X_n-trajectory of (x_1, x_2), as set of points, equals the open half-line spanned by the vector (x_1, x_2) and hence one of its limits is the origin and the other one the infinity.

It is easily checked that the set of $(x_1, x_2) \in \mathbb{R}^2$ such that $(x_1 + ix_2)^{n-1} \in \mathbb{R}$ consists of $n - 1$ vector lines each of them including two exceptional trajectories. These lines regarded as elements of $\mathbb{R}P^1 = \tilde{p}^{-1}(0)$ are the primary singularities of $\widetilde{X}_n$.

AMS Subject Classification: 20F16, 58J20, 37F75, 37O25, 54H25.

Key words: Vector field. Singularity. Zero set. Essential block. Surface.

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(M.W. Hirsch) Department of Mathematics University of Wisconsin at Madison, University of California at Berkeley

E-mail address, M.W. Hirsch: `mwhirsch@chorus.net`

(F.J. Turiel) Geometría y Topología, Facultad de Ciencias, Campus de Teatinos, s/n, 29071-Málaga, Spain

E-mail address, F.J. Turiel: `turiel@uma.es`